Journal of Engineering Mathematics 20 (1986) 27–50

On the Kutta condition for the flow along a semi-infinite elastic plate

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(Received June 13, 1985)

Summary

In this paper we consider a semi-infinite elastic plate, placed in a parallel flow, which performs a waving motion induced by the fluid. We discuss some aspects of the possibility of a smooth flow at the trailing edge. Numerical calculations have been made for a physically realistic part of the parameter space.

1. Introduction

In steady and unsteady linearized profile theory it is generally assumed that the flow is smooth at the trailing edge of a profile. This means that no pressure jump or infinite velocities occur at this edge. In other words, it is assumed that we satisfy the Kutta-condition by which the mathematical problem has a unique solution. In fact we can always add an arbitrary circulation to the flow around a profile of which the shape, possibly as a function of time, is prescribed.

The shape of an elastic profile, however, cannot be prescribed because it is flexible and gives way to the pressures. The addition of a circulation changes the pressures and by this the camber of the profile. Hence it is not certain that a smooth flow at the trailing edge can be obtained.

This hydro-elastic phenomenon can occur, for instance, at the trailing edge of a blade of a ship's screw. Such a blade can carry out elastic vibrations which cause the "singing" of a screw. Questions are: can these vibrations be described by waves in the blade and, if this is true, can the Kutta condition be satisfied for the waving motion of the elastic profile?

In order to obtain some insight in this problem we discuss a simple two-dimensional model for which the behaviour of the flow at the trailing edge can be determined analytically. It consists of a semi-infinite plate of constant bending stiffness and mass distribution per unit of area. It is placed in an incompressible and inviscid flow parallel to the plate, perpendicular to the edge of the plate, which is the trailing edge.

By solving a Wiener-Hopf problem, conditions have been derived for the flow to be smooth at the trailing edge. Then it has to be investigated by numerical means if these conditions can be satisfied. This has been done in this paper for only a part of the set of possible parameters for which mechanically reasonable motions occur. It has been shown that for these values the above-mentioned conditions can not be satisfied. More numerical work can be carried out in the future for the remaining values of the parameters although the physical meaning of a configuration with other parameter values is not so clear. Anyhow, this is a laborious task. The required numerical integrations are quite complicated because of the occurrence of poles on the lines of the integration in different configurations for different sets of parameters. Finally, we mention that the analogous problem for a semi-infinite membrane has been formulated in [1].

2. The two-sided infinite plate

A Cartesian coordinate system $\bar{x}$, $\bar{y}$, $\bar{z}$ is embedded in a fluid with density ρ flowing with a uniform velocity U in the negative $\bar{x}$ direction.

In this flow is placed a two-sided infinite plate, which in its undisturbed position coincides with the plane $\bar{y}=0$. In the following we assume the phenomena to be independent of the $\bar{z}$ coordinate, so the problem is a two-dimensional one (Figure 2.1.). Also we assume the deviations of the plate from the plane $\bar{y}=0$ to be small with respect to occurring wavelengths and we develop a linear theory.

The fluid will be, in the first instance, compressible and endowed with a bulk viscosity determined by the coefficient of expansive friction $\bar{\epsilon}$ [2]. By introducing this type of viscosity the flow remains irrotational. Then the velocities induced in the fluid by its interaction with the plate can be derived from a disturbance potential $\bar{\varphi}(\bar{x}, \bar{y}, \bar{t})$, in which $\bar{t}$ represents time. The linearized equation for this potential becomes [2]

$$\left(\frac{\partial^2}{\partial \bar{x}^2}+\frac{\partial^2}{\partial \bar{y}^2}\right)\left(\bar{\varphi}-\frac{\bar{\epsilon} U}{\rho c^2}\frac{\partial \bar{\varphi}}{\partial \bar{x}}+\frac{\bar{\epsilon}}{\rho c^2}\frac{\partial \bar{\varphi}}{\partial \bar{t}}\right)-\frac{U^2}{c^2}\frac{\partial^2 \bar{\varphi}}{\partial \bar{x}^2}+\frac{2U}{c^2}\frac{\partial^2 \bar{\varphi}}{\partial \bar{x}\,\partial \bar{t}}-\frac{1}{c^2}\frac{\partial^2 \bar{\varphi}}{\partial \bar{t}^2}=0, \quad (2.1)$$

where c is the velocity of sound.

The displacement of the plate in the $\bar{y}$ direction is denoted by $\bar{w}(\bar{x}, \bar{t})$, its flexural rigidity by $\bar{D}$ and its mass per unit area by $\bar{m}$. The linearized boundary condition for $\bar{\varphi}$ at the plate can be formulated as

$$\frac{\partial \bar{\varphi}^-}{\partial \bar{y}}=\frac{\partial \bar{\varphi}^+}{\partial \bar{y}}=\frac{\partial \bar{w}}{\partial \bar{t}}-U\frac{\partial \bar{w}}{\partial \bar{x}}, \quad \bar{y}=0, \quad (2.2)$$

in which $\bar{\varphi}^+$ and $\bar{\varphi}^-$ are the limiting values of $\bar{\varphi}$ from above ($\bar{y}>0$) and from below

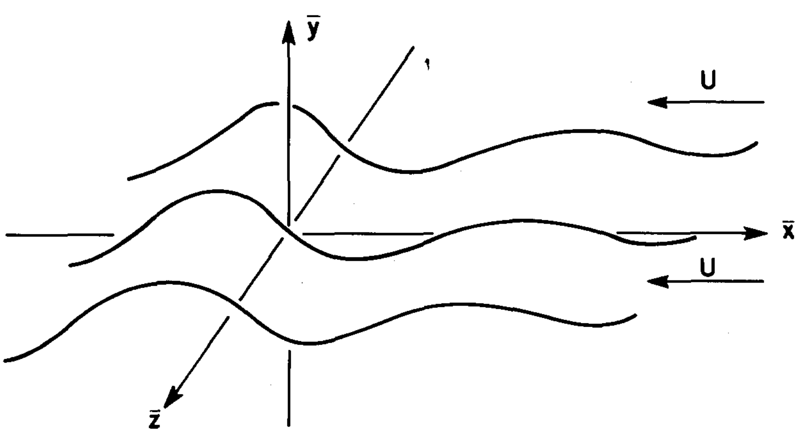

Figure 2.1. The waving motion of a two-sided infinite plate.

($\bar{y} < 0$), respectively (Figure 2.1). By use of the unsteady Bernoulli equation the equation of motion of the plate becomes

$$\bar{m}\frac{\partial^2\bar{w}}{\partial\bar{t}^2} = -\bar{D}\frac{\partial^4\bar{w}}{\partial\bar{x}^4} - \bar{\eta}\frac{\partial\bar{w}}{\partial\bar{t}} - \rho\left(\frac{\partial\bar{\varphi}^-}{\partial\bar{t}} - U\frac{\partial\bar{\varphi}^-}{\partial\bar{x}}\right) + \rho\left(\frac{\partial\bar{\varphi}^+}{\partial\bar{t}} - U\frac{\partial\bar{\varphi}^+}{\partial\bar{x}}\right), \quad \bar{y} = 0, \quad (2.3)$$

where $\bar{\eta}$ is a damping coefficient. In fact, we assume that the plate is coupled to its neutral position (the plane $\bar{y} = 0$) by means of continuously distributed dashpots. Then we have to solve equation (2.1) subject to conditions (2.2) and (2.3).

We assume harmonic time dependence,

$$\bar{\varphi}(\bar{x}, \bar{y}, \bar{t}) = \bar{\varphi}(\bar{x}, \bar{y})\, e^{-i\omega\bar{t}} \quad (2.4)$$

and

$$\bar{w}(\bar{x}, \bar{t}) = \bar{w}(\bar{x})\, e^{-i\omega\bar{t}}. \quad (2.5)$$

On account of "symmetry" we have

$$\bar{\varphi}(\bar{x}, \bar{y}) = -\bar{\varphi}(\bar{x}, -\bar{y}). \quad (2.6)$$

Substitution of (2.4), (2.5) and (2.6) into (2.1), (2.2) and (2.3) yields for $\bar{\varphi}(\bar{x}, \bar{y})$ the equation

$$\left(\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2}\right)\left(\bar{\varphi} - \frac{\bar{\epsilon}U}{\rho c^2}\frac{\partial\bar{\varphi}}{\partial\bar{x}} - \frac{i\omega\bar{\epsilon}}{\rho c^2}\bar{\varphi}\right) - \frac{U^2}{c^2}\frac{\partial^2\bar{\varphi}}{\partial\bar{x}^2} - 2\frac{i\omega U}{c^2}\frac{\partial\bar{\varphi}}{\partial\bar{x}} + \frac{\omega^2}{c^2}\bar{\varphi} = 0, \quad (2.7)$$

with boundary conditions

$$-i\omega\bar{w} - U\frac{\partial\bar{w}}{\partial\bar{x}} = \frac{\partial\bar{\varphi}^+}{\partial\bar{y}}, \quad \bar{y} = 0, \quad (2.8)$$

and

$$-\bar{m}\omega^2\bar{w} + \bar{D}\frac{\partial^4\bar{w}}{\partial\bar{x}^4} - i\bar{\eta}\omega\bar{w} = -2\rho\left(i\omega\bar{\varphi}^+ + U\frac{\partial\bar{\varphi}^+}{\partial\bar{x}}\right), \quad \bar{y} = 0. \quad (2.9)$$

We now introduce the dimensionless quantities

$$x = \bar{x}\omega U^{-1}, \quad y = \bar{y}\omega U^{-1}, \quad w = \bar{w}\omega U^{-1}, \quad t = \bar{t}\omega, \quad m = \bar{m}\omega U^{-1}\rho^{-1}, \quad D = \bar{D}\omega^3 U^{-5}\rho^{-1},$$

$$\varphi = \bar{\varphi}\omega U^{-2}, \quad \epsilon = \bar{\epsilon}\omega U^{-2}\rho^{-1}, \quad \eta = \bar{\eta}U^{-1}\rho^{-1}, \quad N = U^2/c^2. \quad (2.10)$$

Then the boundary-value problem becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\varphi - \epsilon N\frac{\partial\varphi}{\partial x} - i\epsilon N\varphi\right) - N\frac{\partial^2\varphi}{\partial x^2} - 2iN\frac{\partial\varphi}{\partial x} + N\varphi = 0, \quad y > 0, \quad (2.11)$$

$$-iw - \frac{\partial w}{\partial x} = \frac{\partial\varphi^+}{\partial y}, \quad y = 0, \quad (2.12)$$

$$-mw + D\frac{\partial^4 w}{\partial x^4} - \mathrm{i}\eta w = -2\left(\mathrm{i}\varphi^+ + \frac{\partial \varphi^+}{\partial x}\right), \quad y = 0. \tag{2.13}$$

In the following we are interested in the limit of the solution for N and η tending to zero in one way or another which will be discussed in Section 4. It is not necessary to consider also the limit case of a vanishing coefficient of bulk viscosity ϵ because, when N tends to zero, the velocity of sound becomes infinite, so the fluid will become incompressible. The equation for the disturbance potential of the fluid (2.11) then reduces to the Laplace equation.

We now try to find the displacement of the plate and the disturbance potential in the form

$$w(x) = A\,\mathrm{e}^{-\mathrm{i}\lambda x} \tag{2.14}$$

and

$$\varphi(x, y) = \mathrm{e}^{-\{\lambda^2 - N(\lambda-1)^2/(1+\mathrm{i}\epsilon N(\lambda-1))\}^{1/2} y - \mathrm{i}\lambda x} \overset{\mathrm{def}}{=} \mathrm{e}^{-\{B(\lambda)\}^{1/2} y - \mathrm{i}\lambda x}, \quad y > 0, \tag{2.15}$$

with

$$\mathrm{Re}\{B(\lambda)\}^{1/2} = \mathrm{Re}\left\{\lambda^2 - \frac{N(\lambda-1)^2}{1 + \mathrm{i}\epsilon N(\lambda-1)}\right\}^{1/2} > 0, \tag{2.16}$$

where A is still an arbitrary constant. Then $\varphi(x, y)$ satisfies equation (2.11) and, by (2.6), vanishes for $y \to \pm\infty$ for all values of λ. Writing the argument of the square root (2.16) in the form

$$B(\lambda) = \frac{\lambda^2\{1 + \mathrm{i}\epsilon N(\lambda-1)\} - N(\lambda-1)^2}{1 + \mathrm{i}\epsilon N(\lambda-1)}, \tag{2.17}$$

we see there are three zeroes (λ_1, λ_2 and λ_3) of the numerator and one of the denominator (λ_4). These four points are the branch points of the square root and are given by

$$\begin{aligned}
\lambda_1 &= N^{1/2} - N + \left(1 + \frac{\mathrm{i}\epsilon}{2}\right)N^{3/2} + \mathrm{O}(N^2),\\
\lambda_2 &= -N^{1/2} - N - \left(1 + \frac{\mathrm{i}\epsilon}{2}\right)N^{3/2} + \mathrm{O}(N^2),\\
\lambda_3 &= 1 + \frac{\mathrm{i}}{\epsilon}\left(\frac{1}{N} - 1\right) + \mathrm{O}(N),\\
\lambda_4 &= 1 + \frac{\mathrm{i}}{\epsilon N}.
\end{aligned} \tag{2.18}$$

In the complex λ-plane we can find precisely three lines on which $B(\lambda)$ is real and negative. One of these lines starts at λ_1 and runs towards infinity, essentially along the imaginary axis in the upper halfplane; another starts at λ_2 and runs towards infinity

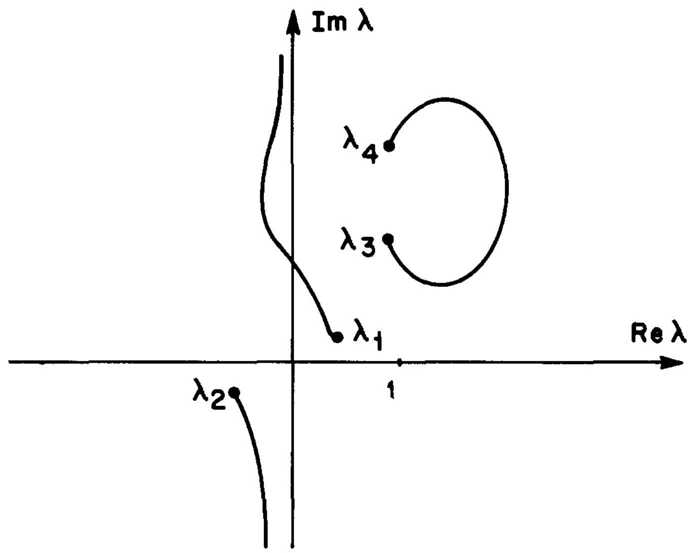

Figure 2.2. The complex λ-plane with the cuts.

essentially along the imaginary axis but now in the lower halfplane. Finally we have a line connecting λ_3 and λ_4.

We introduce these three lines as cuts in the complex λ-plane (Figure 2.2). Then the argument of the complex quantity $B(\lambda)$ stays within the interval $(-\pi, \pi)$. We define the square root $(B(\lambda))^{1/2}$ to be equal to 1 for $\lambda = 1$. Then, first, the root has become single-valued and, second, (2.16) is satisfied for all values of λ.

Substitution of (2.14) and (2.15) into (2.12) and (2.13) yields

$$A(\mathrm{i} - \mathrm{i}\lambda) = \{B(\lambda)\}^{1/2} \tag{2.19}$$

and

$$A(m + \mathrm{i}\eta - \lambda^4 D) = 2(\mathrm{i} - \mathrm{i}\lambda), \tag{2.20}$$

respectively. Hence λ has to satisfy the equation for the wavelength

$$\{B(\lambda)\}^{1/2}(m + \mathrm{i}\eta - \lambda^4 D) = -2(1-\lambda)^2, \tag{2.21}$$

where, by (2.16), the real part of the square root has to be positive.

3. The equation for the wavelength

Temporarily, we set N and η equal to zero. Then the equation for the wavelength (2.21) changes into

$$\{\lambda^2\}^{1/2}(m - \lambda^4 D) = -2(1-\lambda)^2, \qquad \mathrm{Re}\{\lambda^2\}^{1/2} > 0. \tag{3.1}$$

We define

$$f(\lambda) = D\lambda^5 + 2\lambda^2 - (m+4)\lambda + 2, \qquad \text{Re}\,\lambda < 0, \tag{3.2}$$

and

$$g(\lambda) = D\lambda^5 - 2\lambda^2 - (m-4)\lambda - 2, \qquad \text{Re}\,\lambda > 0. \tag{3.3}$$

Then the roots of (3.1) are the zeroes of $f(\lambda)$ and $g(\lambda)$ subject to the conditions Re $\lambda < 0$ and Re $\lambda > 0$, respectively.

First we consider $f(\lambda)$, hence Re $\lambda < 0$. It is seen from Figure 3.1 that when λ travels along the contour indicated in Figure 3.1a we will have by the principle of the argument [3] one zero of $f(\lambda)$ when $D - m - 4 < 0$ and three zeroes when $D - m - 4 > 0$, both cases with Re $\lambda < 0$.

Since the coefficients of (3.2) are real, the zero of $f(\lambda)$ for $D - m - 4 < 0$ is real. When $D - m - 4 > 0$, we have one real and two complex conjugate zeroes. Indeed, let us assume the three zeros to be real. Then the first derivative of $f(\lambda)$ must have two negative real zeroes. However, its argument increases only by 2π when λ travels along the contour of Figure 3.1a. Hence the first derivative of $f(\lambda)$ possesses only one zero with Re $\lambda < 0$ which contradicts our assumption.

Next we consider the function $g(\lambda)$, hence Re $\lambda > 0$ and suppose $4 > m$. The number of revolutions of $g(\lambda)$ around the origin when λ travels along the contour of Figure 3.2a is easily seen to be one for $D - m + 4 < 0$ and three for $D - m + 4 > 0$ (Figure 3.2b). For the case of three zeroes ($D - m + 4 > 0$) an analogous reasoning as before shows that one zero is real and two zeroes are complex conjugate.

Now let us discuss $g(\lambda)$, (Re $\lambda > 0$), when $m \leqslant 4$. For λ travelling along the contour of Figure 3.2a, the corresponding contour is given in Figure 3.3. Hence, in this case, there are three zeroes with Re $\lambda > 0$.

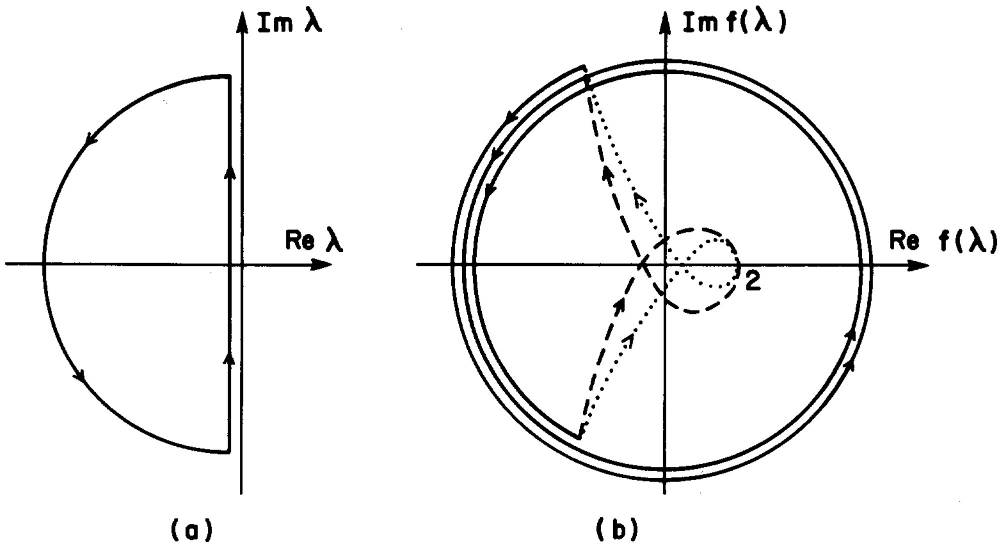

Figure 3.1. The number of zeroes of $f(\lambda)$, (3.2), $D - m - 4 < 0$: - - - - - -, $D - m - 4 > 0$: $\cdots\cdots$.

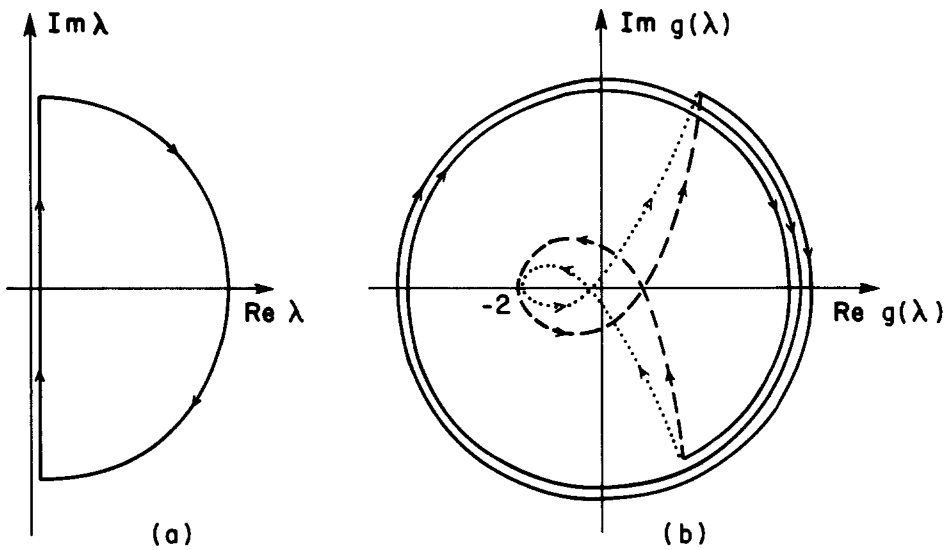

Figure 3.2. The number of zeroes of $g(\lambda)$, (3.3), for $m > 4$, $D - m + 4 < 0$: - - - - - -, $D - m + 4 > 0$: · · · · · ·.

We now investigate the case of three real zeroes. The function $g(\lambda)$ possesses a zero of order two if

$$D\lambda^5 - 2\lambda^2 - (m-4)\lambda - 2 = 5D\lambda^4 - 4\lambda - (m-4) = 0. \tag{3.4}$$

Elimination of D yields two zeroes of order two, $\lambda = \nu_1$ and $\lambda = \nu_2$, for the function $g(\lambda)$:

$$\nu_1 = \frac{4 - m - \{m^2 - 8m + 1\}^{1/2}}{3} \tag{3.5}$$

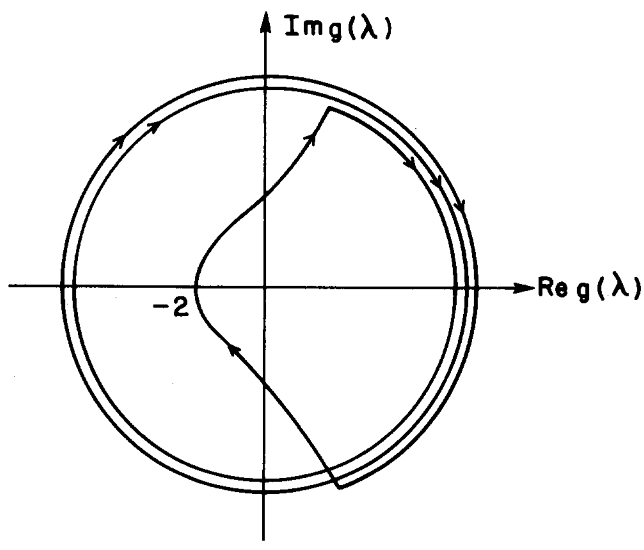

Figure 3.3. The number of zeroes of $g(\lambda)$, (3.3), for $m \leqslant 4$.

and

$$\nu_2 = \frac{4 - m + \{m^2 - 8m + 1\}^{1/2}}{3}. \tag{3.6}$$

In order that these zeroes be real (and hence positive, see Fig. 3.4), we must have instead of $m \leqslant 4$ the stronger condition

$$m < 4 - \sqrt{15}. \tag{3.7}$$

Then we have

$$1 < \nu_1 < \nu_2, \tag{3.8}$$

while the corresponding values of D are given by

$$D = \frac{4\nu_1 + m - 4}{5\nu_1^4} \tag{3.9}$$

and

$$D = \frac{4\nu_2 + m - 4}{5\nu_2^4} > \frac{4\nu_1 + m - 4}{5\nu_1^4}. \tag{3.10}$$

The inequality sign in (3.10) follows from straightforward calculation.

First we suppose (3.7) to be satisfied. Since the derivative of $g(\lambda)$ with respect to D is positive for positive λ, we have three real zeroes for

$$\frac{4\nu_1 + m - 4}{5\nu_1^4} < D < \frac{4\nu_2 + m - 4}{5\nu_2^4}, \tag{3.11}$$

and one real zero for the other values of D.

Next we discuss the remaining values for m, namely $4 - \sqrt{15} < m \leqslant 4$. Suppose there exist three real zeroes for some values of D. Then from the special form of $g(\lambda)$ it follows that by increasing D there must arise a real zero of order two of $g(\lambda)$. However, for the

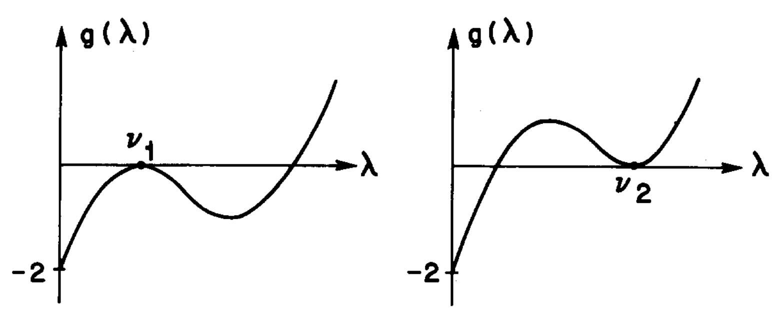

Figure 3.4. The real zeroes of order two of $g(\lambda)$ for different values of D.

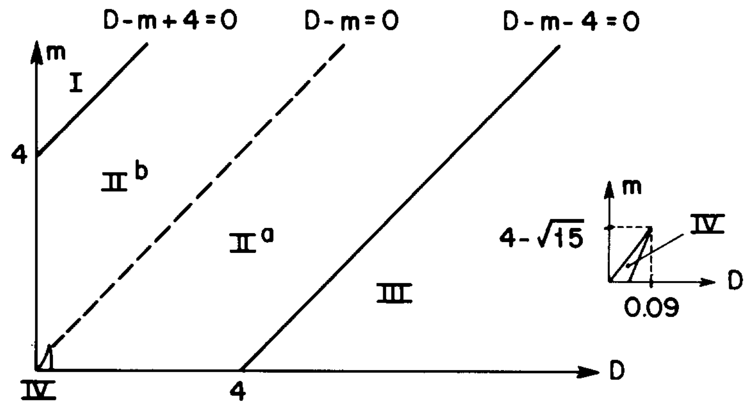

Figure 3.5. The partition of the D, m-plane with respect to the number and types of the roots of (3.1).

values of m under discussion ($4-\sqrt{15} < m \leqslant 4$) there does not exist a real zero of order two of $g(\lambda)$ for any value of D on account of (3.7). Hence we have one real zero and two complex conjugate zeroes for all values of D when $4-\sqrt{15} < m \leqslant 4$.

In order to give a survey of the discussion above, we divide the D, m-plane into regions (Figure 3.5.) which are of significance for the number and types of the roots of (3.1):

Region I, with $D-m+4<0$,
Region III, with $D-m-4>0$,
Region IV, with

$$\frac{4\nu_1+m-4}{5\nu_1^4} < D < \frac{4\nu_2+m-4}{5\nu_2^4}, \tag{3.12}$$

Region II, with $D-m+4>0$ and $D-m-4<0$, minus region IV.

For these regions we have the following results:

Region I, two roots, one positive and one negative,
Region II, four roots, one positive, one negative and two complex conjugate with $\operatorname{Re}\lambda>0$,
Region III, six roots, one positive, one negative, two complex conjugate with $\operatorname{Re}\lambda>0$ and two complex conjugate with $\operatorname{Re}\lambda<0$,
Region IV, four roots, three positive and one negative.

The values of m and D on the boundary of the four regions belong to a set of measure zero and for that reason they will not be considered. We only remark that for values of m and D with $D-m\pm 4=0$ standing waves occur ($\operatorname{Re}\lambda=0$).

4. The influence of the parameters η and N

In Section 3 we considered the equation for the wavelength (2.20) with η and N equal to zero. We will now discuss the influence of small values of η and N on the character of the waves.

We assumed in Section 2 by (2.5), (2.10) and (2.14):

$$w(x, t) = A\,\mathrm{e}^{-\mathrm{i}(\lambda x + t)}, \tag{4.1}$$

where λ has to be a root of (2.21). For Re $\lambda > 0$ we have a wave moving in the downstream direction and for Re $\lambda < 0$ a wave moving upstream. The sign of the imaginary part of λ indicates that we have a wave with an increasing or a decreasing amplitude in its direction of motion. When the imaginary part is zero we have a wave with constant amplitude. Hence for η and N equal to zero we have the following waves for the regions denoted in Figure 3.5:

Region I, two waves, one downstream and one upstream, both of constant amplitude,
Region II, four waves, one downstream and one upstream, both of constant amplitude, and two downstream, one with increasing and one with decreasing amplitude,
Region III, six waves, one downstream and one upstream, both of constant amplitude; further two downstream and two upstream, each type has one wave with increasing and one with decreasing amplitude,
Region IV, four waves, three downstream and one upstream, all of constant amplitude.

The waves increasing or decreasing for $\eta = N = 0$, increase or decrease exponentially. They will continue to increase or decrease exponentially for sufficiently small values of η and N. We will call them strongly-increasing or strongly-decreasing, respectively. Whether the waves of constant amplitude (λ real) for $\eta = N = 0$ become increasing or decreasing for non-zero values of η and N depends on the influence of η and N on the real zeroes λ of (3.1). In fact, when a root is displaced from the real line into the upper (lower) halfplane it means that the amplitude of the wave increases (decreases) with increasing values of x.

We now calculate the imaginary part of the disturbed values of λ for those waves which in the inviscid case ($\eta = N = 0$) have constant amplitude. We denote the disturbed roots by $\tilde{\lambda}$ and write $\tilde{\lambda} = \lambda_r + \gamma$ where λ_r is the real root for the inviscid case. Substitution of $\tilde{\lambda} = \lambda_r + \gamma$ in (2.21) and expansion of the resulting equation with respect to small values of γ, η and N yields

$$\mathrm{Im}\,\tilde{\lambda} = \mathrm{Im}\,\gamma = \frac{\lambda_r}{h(\lambda_r)}\left\{-|\lambda_r|\eta + \epsilon N^2 \frac{(\lambda_r - 1)^5}{\lambda_r^2} + \mathrm{O}(\eta N) + \mathrm{O}(N^3)\right\}, \tag{4.2}$$

where

$$h(\lambda_r) = -6\lambda_r^2 + 16\lambda_r - 10 - 4|\lambda_r|m. \tag{4.3}$$

We consider the two cases $\lambda_r < 0$ and $\lambda_r > 0$ separately.

First, $\lambda_r < 0$; then we have upstream waves. Because of $h(\lambda)$ being negative for $\lambda < 0$, Im γ will be negative for all small values of η and N. This means that the corresponding upstream waves for $\eta > 0$ and $N > 0$ have a slowly-decreasing amplitude. By a slowly-decreasing (increasing) amplitude we mean that the decreasing (increasing) behaviour disappears in the inviscid and incompressible case, that is, for $\eta = N = 0$.

Second, $\lambda_r > 0$; then we have downstream waves. In this case the function $h(\lambda)$ is positive if and only if

$$m < 4 - \sqrt{15} \tag{4.4}$$

and

$$\nu_1 < \lambda < \nu_2, \tag{4.5}$$

where ν_1 and ν_2 are defined in (3.5) and (3.6), respectively. Hence $h(\lambda_r)$ is positive for the middle one of the three positive real zeroes in region IV and negative for all other positive real zeroes in the whole D, m-plane.

Since $g(0) = -2$ and $g(1) = D - m$, the positive real zero for m and D in the regions I, II and III is smaller (greater) than one for $m < D$ ($m > D$). This means that for $m < D$ in the regions II and III Im λ will be positive for all small values of η and N. So we have a wave with a slowly-decreasing amplitude. For $m > D$ in the regions I and II the sign of Im γ depends on the way in which η and N tend to zero. We have a wave with a slowly-decreasing or increasing amplitude.

When m and D are situated in region IV there are three positive real zeroes λ_{r1}, λ_{r2} and λ_{r3} with

$$\lambda_{r1} < \nu_1 < \lambda_{r2} < \nu_2 < \lambda_{r3}. \tag{4.6}$$

On account of (3.8), λ_{r2} and λ_{r3} are greater than one. For $m < D$ ($m > D$) λ_{r1} will be smaller (greater) than one. Hence the sign of Im γ_1, when $m > D$, and the signs of Im γ_2 and Im γ_3 depend on the values of η and N. For $m < D$ Im γ_1 will be positive for all values of η and N.

So we find that some waves of constant amplitude for $\eta = N = 0$ can be the limit of an increasing or of a decreasing wave. This depends on the behaviour of η/N for $\eta \to 0$ and $N \to 0$. In other words, from one such a wave ($\eta = N = 0$) can arise a slowly-increasing or slowly-decreasing wave for $\eta \neq 0$, $N \neq 0$. Only for $m < D$ in the regions II and III the behaviour of the upstream and downstream waves, which in the inviscid and incompressible case are waves of constant amplitude, is unique. They are waves with a slowly-decreasing amplitude.

In the next section we will discuss the waving motion of a semi-infinite plate. At great distance from the trailing edge we expect the same type of waves as in the case of the two-sided infinite plate. We want to restrict ourselves to values of m and D for which the above-mentioned unique behaviour of the slowly-decreasing waves occurs. We assume the plate to be at rest at $x = +\infty$ in the case of non-zero values of η and N. Hence m and D will be chosen in region II with $m < D$. This region will be denoted by II^{a}. In the sequel we will restrict ourselves to this region.

5. The semi-infinite plate

In the following sections we will investigate the fluid flow in the neighbourhood of the trailing edge of the plate. We assume the leading edge to be upstream at infinity, since it is to be expected that the nature of a possible singular behaviour of the flow depends only on what happens in the direct neighbourhood of the trailing edge.

Using the same dimensionless quantities as introduced in (2.10) we find the following boundary-value problem for the semi-infinite plate:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\varphi - \epsilon N\frac{\partial \varphi}{\partial x} - \mathrm{i}\epsilon N\varphi\right) - N\frac{\partial^2 \varphi}{\partial x^2} - 2\mathrm{i}N\frac{\partial \varphi}{\partial x} + N\varphi = 0, \tag{5.1}$$

$$\frac{\partial\varphi^+}{\partial y}=\frac{\partial\varphi^-}{\partial y},\qquad \mathrm{i}\varphi^+ + \frac{\partial\varphi^+}{\partial x}=\mathrm{i}\varphi^- + \frac{\partial\varphi^-}{\partial x},\quad x<0,\ y=0, \tag{5.2}$$

$$\frac{\partial\varphi^+}{\partial y}=\frac{\partial\varphi^-}{\partial y}=-\mathrm{i}w-\frac{\partial w}{\partial x},\quad x>0,\ y=0, \tag{5.3}$$

$$\left(\mathrm{i}\varphi^+ + \frac{\partial\varphi^+}{\partial x}\right)-\left(\mathrm{i}\varphi^- + \frac{\partial\varphi^-}{\partial x}\right)=-D\frac{\partial^4 w}{\partial x^4}+mw+\mathrm{i}\eta w,\quad x>0,\ y=0. \tag{5.4}$$

We now introduce the Fourier transforms

$$\phi_-(\lambda,\ y\gtrless 0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\varphi(x,\ y\gtrless 0)\,\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x, \tag{5.5}$$

$$\phi_+(\lambda,\ y\gtrless 0)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\varphi(x,\ y\gtrless 0)\,\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x, \tag{5.6}$$

$$\phi(\lambda,\ y\gtrless 0)=\phi_-(\lambda,\ y\gtrless 0)+\phi_+(\lambda,\ y\gtrless 0). \tag{5.7}$$

For m and D in the region II^a of Figure (3.5) and η and N non-zero, we have an upstream wave in the plate and in the fluid with a slowly-decreasing amplitude. The downstream wave in this region has a strongly-increasing amplitude, hence the "leading edge" at $x=+\infty$ is at rest. By this motion of the plate free vorticity is shed at the trailing edge. This concentrated free-vortex sheet is not dispersed in the fluid even when η and $N\neq 0$, hence theoretically its strength does not tend to zero for $x\to-\infty$. This can be understood by the absence of shearing forces in our model of the fluid. Then the disturbance velocities of the fluid in the wake of the plate do not vanish for $x\to-\infty$. Hence in order to apply a Fourier transformation to equations (5.1)–(5.4) we have to take the imaginary part of λ negative and with sufficiently small absolute value.

By transformation of (5.1) we obtain

$$\phi(\lambda,\ y>0)=E(\lambda)\,\mathrm{e}^{-\{\lambda^2-N(\lambda-1)^2/(1+\mathrm{i}\epsilon N(\lambda-1))\}^{1/2}y} \tag{5.8}$$

and

$$\phi(\lambda,\ y<0)=F(\lambda)\,\mathrm{e}^{\{\lambda^2-N(\lambda-1)^2/(1+\mathrm{i}\epsilon N(\lambda-1))\}^{1/2}y}, \tag{5.9}$$

with again (2.16),

$$\mathrm{Re}\left\{\lambda^2-\frac{N(\lambda-1)^2}{1+\mathrm{i}\epsilon N(\lambda-1)}\right\}^{1/2}>0, \tag{5.10}$$

for all values of λ. The cuts in the complex λ-plane, introduced to make the square root single-valued, will be the same as in Section 2.

From (5.2) and (5.3) we find

$$\frac{\partial\phi}{\partial y}(\lambda,0^+)=\frac{\partial\phi}{\partial y}(\lambda,0^-). \tag{5.11}$$

Hence, we must have on account of (5.8) and (5.9):

$$E(\lambda) = -F(\lambda). \tag{5.12}$$

Fourier transformation of (5.3) yields

$$\frac{\partial \phi_+}{\partial y}(\lambda, 0^+) = \mathrm{i}(\lambda - 1)W_+(\lambda) + \frac{w(0)}{\sqrt{2\pi}}, \tag{5.13}$$

where $W_+(\lambda)$ is the one-sided Fourier transform of $w(x)$ for $0 < x < \infty$, analogous to (5.6). From (5.2) and (5.4) we obtain

$$\begin{aligned} &\mathrm{i}(1-\lambda)\{\phi(\lambda, 0^+) - \phi(\lambda, 0^-)\} \\ &\quad = (m + \mathrm{i}\eta - \lambda^4 D)W_+(\lambda) - \frac{\lambda^2 D}{\sqrt{2\pi}}\frac{\partial w}{\partial x}(0) + \frac{\mathrm{i}\lambda^3 D}{\sqrt{2\pi}} w(0), \end{aligned} \tag{5.14}$$

where we supposed $\partial^2 w/\partial x^2(0)$ and $\partial^3 w/\partial x^3(0)$ to be zero. This means that we do not expect such a strong singularity of the pressure in the neighbourhood of the trailing edge of the plate that it would induce a singular normal force in the y-direction or a singular bending moment. In fact, under these assumptions it turns out that the singularity of the pressure near the trailing edge is at most of strength $\mathrm{O}(x^{-1/2})$, which does not give rise to singular forces or moments at $x = 0$.

Combination of equations (5.7) upto (5.14) results in

$$\begin{aligned} &\frac{\partial \phi_-}{\partial y}(\lambda, 0) - \frac{\mathrm{i}}{2(1-\lambda)} W_+(\lambda)\left[2(1-\lambda)^2 + (m + \mathrm{i}\eta - \lambda^4 D)\right. \\ &\qquad\qquad \left. \times \left\{\lambda^2 - \frac{N(\lambda-1)^2}{1 + \mathrm{i}\epsilon N(\lambda - 1)}\right\}^{1/2}\right] \\ &\quad = \frac{-w(0)}{\sqrt{2\pi}}\left[1 + \frac{\lambda^3 D}{2(1-\lambda)}\left\{\lambda^2 - \frac{N(\lambda-1)^2}{1 + \mathrm{i}\epsilon N(\lambda-1)}\right\}^{1/2}\right] \\ &\qquad - \frac{\mathrm{i}\lambda^2 D}{2\sqrt{2\pi}(1-\lambda)}\frac{\partial w}{\partial x}(0)\left\{\lambda^2 - \frac{N(\lambda-1)^2}{1 + \mathrm{i}\epsilon N(\lambda-1)}\right\}^{1/2} \end{aligned} \tag{5.15}$$

This Hilbert problem for the unknown functions $\partial\phi_-/\partial y$ and W_+ is defined on a line L just below the real axis of the complex λ-plane. The line L divides the plane into regions S^+ and S^-, situated to the left- and right-hand side of L with respect to the positive x-direction. In general, a subscript "+" or "−" attached to a function denotes that such a function is regular in S^+ or S^-, respectively.

6. The Hilbert problem

First we solve the homogeneous part of equation (5.15) in a way as discussed in [4]. Writing

$$X_-(\lambda) - X_+(\lambda)\frac{\mathrm{i}}{2(1-\lambda)}\left[2(1-\lambda)^2 + \left(m + \mathrm{i}\eta - \lambda^4 D\right)\left\{\lambda^2 - \frac{N(\lambda-1)^2}{1+\mathrm{i}\epsilon N(\lambda-1)}\right\}^{1/2}\right]$$

$$\overset{\text{def}}{=} X_-(\lambda) - X_+(\lambda)T(\lambda) = 0, \tag{6.1}$$

we have to perform the factorization of $T(\lambda)$. This is done by multiplying $T(\lambda)$ by a suitable function in such a way that the resulting function $G(\lambda)$ has the following properties:

1) $G(\lambda)$ satisfies the Hölder condition on the line L,
2) $G(\lambda)$ has no zeroes on the line L,
3) $G(\lambda)$ tends to one for $\mathrm{Re}\,\lambda \to \pm\infty$ on L,
4) The increase of the argument of $G(\lambda)$, when λ travels along L from $-\infty$ to $+\infty$, is zero.

The asymptotic behaviour of $T(\lambda)$ on L reads

$$\lim_{\mathrm{Re}\,\lambda \to \pm\infty} T(\lambda) = \pm\frac{\mathrm{i}D}{2}\lambda^4. \tag{6.2}$$

We define $G(\lambda)$ by

$$G(\lambda) = T(\lambda)\frac{1}{(\lambda^2+s^2)^{3/2}(\lambda+\mathrm{i}q)}\frac{2}{\mathrm{i}D}, \tag{6.3}$$

where s is some real positive number and where the square root is made single-valued by requiring

$$\mathrm{Re}\{\lambda^2+s^2\}^{1/2} > 0. \tag{6.4}$$

The sign of the constant q will be discussed in the next section. In connection with the required property (4) of $G(\lambda)$ we will find that q has to be positive. Then $G(\lambda)$ has the properties (1) upto and including (4).

Now we can factorize $G(\lambda)$,

$$G(\lambda) = \frac{G_-(\lambda)}{G_+(\lambda)}, \quad \lambda \in L, \tag{6.5}$$

where

$$G_\pm(\lambda) = \exp -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\ln G(\xi)}{\xi-\lambda}\mathrm{d}\xi, \quad \xi \in L,\ \lambda \in S^\pm. \tag{6.6}$$

The integral in (6.6) is convergent on account of the properties of $G(\lambda)$. From (6.1), (6.3) and (6.6) we find ($q > 0$)

$$
\begin{aligned}
X_-(\lambda) &= G_-(\lambda)\frac{\mathrm{i}D}{2}(\lambda - \mathrm{i}s)^{3/2},\\
X_+(\lambda) &= G_+(\lambda)\frac{1}{(\lambda + \mathrm{i}s)^{3/2}(\lambda + \mathrm{i}q)}.
\end{aligned} \tag{6.7}
$$

Substitution of this result into the inhomogeneous equation (5.15) yields

$$
\begin{aligned}
&\frac{\partial \phi_-}{\partial y}(\lambda, 0)\frac{1}{X_-(\lambda)} - W_+(\lambda)\frac{1}{X_+(\lambda)}\\
&\quad = -\frac{w(0)}{\sqrt{2\pi}}\left[1 + \frac{\lambda^3 D}{2(1-\lambda)}\left\{\lambda^2 - \frac{N(\lambda-1)^2}{1 + \mathrm{i}\epsilon N(\lambda - 1)}\right\}^{1/2}\right]\frac{1}{X_-(\lambda)}\\
&\qquad - \frac{\partial w}{\partial x}(0)\frac{\mathrm{i}\lambda^2 D}{2\sqrt{2\pi}(1-\lambda)}\left\{\lambda^2 - \frac{N(\lambda-1)^2}{1 + \mathrm{i}\epsilon N(\lambda - 1)}\right\}^{1/2}\frac{1}{X_+(\lambda)}.
\end{aligned} \tag{6.8}
$$

Now we have to split the right-hand side of (6.8) into two terms, one of which is regular in S^- and the other is regular in S^+. By use of (6.1) this right-hand side can be written as

$$
\begin{aligned}
&-\frac{w(0)}{\sqrt{2\pi}\,X_-(\lambda)} - \frac{1}{\sqrt{2\pi}}\left\{w(0)\lambda^3 D + \mathrm{i}\lambda^2 D\frac{\partial w}{\partial x}(0)\right\}\\
&\times\left\{\frac{-\mathrm{i}}{X_+(\lambda)} - \frac{(1-\lambda)}{X_-(\lambda)}\right\}\frac{1}{(m + \mathrm{i}\eta - \lambda^4 D)}.
\end{aligned} \tag{6.9}
$$

By adding and subtracting poles we obtain

$$
\begin{aligned}
&\left[\frac{-w(0)}{\sqrt{2\pi}\,X_-(\lambda)} - \left\{\frac{w(0)\lambda^3 D + \mathrm{i}\lambda^2 D\frac{\partial w}{\partial x}(0)}{\sqrt{2\pi}}\right\}\left\{\frac{-(1-\lambda)}{(m + \mathrm{i}\eta - \lambda^4 D)X_-(\lambda)}\right.\right.\\
&\quad + \frac{(1-\lambda)}{4(m+\mathrm{i}\eta)^{3/4}\left\{(m+\mathrm{i}\eta)^{1/4} - \mathrm{i}D^{1/4}\lambda\right\}X_-\left(-\mathrm{i}\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)}\\
&\quad + \frac{(1-\lambda)}{4(m+\mathrm{i}\eta)^{3/4}\left\{(m+\mathrm{i}\eta)^{1/4} + D^{1/4}\lambda\right\}X_-\left(-\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)}
\end{aligned}
$$

$$
-\frac{\mathrm{i}}{4(m+\mathrm{i}\eta)^{3/4}\left\{(m+\mathrm{i}\eta)^{1/4}-D^{1/4}\lambda\right\}X_+\left(\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)}
$$

$$
\left.\left.-\frac{\mathrm{i}}{4(m+\mathrm{i}\eta)^{3/4}\left\{(m+\mathrm{i}\eta)^{1/4}+\mathrm{i}D^{1/4}\lambda\right\}X_+\left(\mathrm{i}\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)}\right\}\right]
$$

$$
-\left[\left\{\frac{w(0)\lambda^3 D+\mathrm{i}\lambda^2 D\frac{\partial w}{\partial x}(0)}{\sqrt{2\pi}}\right\}\left\{\frac{-\mathrm{i}}{(m+\mathrm{i}\eta-\lambda^4 D)X_+(\lambda)}\right.\right.
$$

$$
+\frac{\mathrm{i}}{4(m+\mathrm{i}\eta)^{3/4}\left\{(m+\mathrm{i}\eta)^{1/4}-D^{1/4}\lambda\right\}X_+\left(\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)}
$$

$$
+\frac{\mathrm{i}}{4(m+\mathrm{i}\eta)^{3/4}\left\{(m+\mathrm{i}\eta)^{1/4}+\mathrm{i}D^{1/4}\lambda\right\}X_+\left(\mathrm{i}\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)}
$$

$$
-\frac{(1-\lambda)}{4(m+\mathrm{i}\eta)^{3/4}\left\{(m+\mathrm{i}\eta)^{1/4}-\mathrm{i}D^{1/4}\lambda\right\}X_-\left(-\mathrm{i}\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)}
$$

$$
\left.\left.-\frac{(1-\lambda)}{4(m+\mathrm{i}\eta)^{3/4}\left\{(m+\mathrm{i}\eta)^{1/4}+D^{1/4}\lambda\right\}X_-\left(-\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)}\right\}\right]\overset{\mathrm{def}}{=}H_-(\lambda)+H_+(\lambda).
\tag{6.10}
$$

Using (6.10), the relation (6.8) becomes

$$
\frac{\partial\phi_-}{\partial y}(\lambda,0)\frac{1}{X_-(\lambda)}-H_-(\lambda)=W_+(\lambda)\frac{1}{X_+(\lambda)}+H_+(\lambda),
\tag{6.11}
$$

where the left-hand side is regular in S^- while the right-hand side is regular in S^+. For $|\lambda|\to\infty$, $\lambda\in S^-$ this left-hand side behaves algebraically and the right-hand side behaves algebraically for $|\lambda|\to\infty$, $\lambda\in S^+$. Hence by Liouville's theorem, the general solution of the Hilbert problem (5.15) is given by

$$
\frac{\partial\phi_-}{\partial y}(\lambda,0)=\{H_-(\lambda)+Q(\lambda)\}X_-(\lambda),
\tag{6.12}
$$

$$
W_+(\lambda)=\{-H_+(\lambda)+Q(\lambda)\}X_+(\lambda),
\tag{6.13}
$$

where $Q(\lambda)$ is an unknown polynomial.

7. Discussion of the constant q

In this section we show that the constant q, introduced in (6.3), has to be taken positive in order to have property (4) of $G(\lambda)$ (Section 6). We consider the behaviour of $G(\lambda)$ when λ on L, just below the real axis, while Re λ increases from $-\infty$ to $+\infty$. Substitution of

$$\lambda = \mu - \mathrm{i}\delta, \quad -\infty < \mu < \infty, 0 < \delta \ll (\eta, N^2) \tag{7.1}$$

into (6.3) yields

$$G(\mu - \mathrm{i}\delta) = -\left[2(\mu - 1 - \mathrm{i}\delta)^2 + \left\{m + \mathrm{i}\eta - (\mu - \mathrm{i}\delta)^4 D\right\} \times \left\{(\mu - \mathrm{i}\delta)^2 - \frac{N(\mu - 1 - \mathrm{i}\delta)^2}{1 + \mathrm{i}\epsilon N(\mu - 1 - \mathrm{i}\delta)}\right\}^{1/2}\right] \times \frac{\mu - \mathrm{i}\delta - \mathrm{i}q}{D\left\{(\mu - \mathrm{i}\delta)^2 + s^2\right\}^{3/2}\left\{(\mu - \mathrm{i}\delta)^2 + q^2\right\}(\mu - 1 - \mathrm{i}\delta)}, \tag{7.2}$$

where the real part of the square roots has to be chosen positive, again in connection with (2.16) and (6.4). Expansion of the expression between square brackets of (7.2) for small values of η, N and δ gives

$$\begin{aligned} R &\overset{\text{def}}{=} 2(\mu - 1 - \mathrm{i}\delta)^2 + \left\{m + \mathrm{i}\eta - (\mu - \mathrm{i}\delta)^4 D\right\} \cdot \left\{(\mu - \mathrm{i}\delta)^2 - \frac{N(\mu - 1 - \mathrm{i}\delta)^2}{1 + \mathrm{i}\epsilon N(\mu - 1 - \mathrm{i}\delta)}\right\}^{1/2} \\ &= 2(\mu - 1)^2 - 4\delta(\mu - 1)\mathrm{i} + \left\{m + \mathrm{i}\eta - (\mu^4 - 4\delta\mu^3\mathrm{i})D\right\} \\ &\quad \times |\mu| \left[1 - \frac{\delta\mathrm{i}}{\mu} - \frac{N}{2\mu^2}(\mu^2 - 2\mu + 1 - 2\delta\mu\mathrm{i} + 2\delta\mathrm{i})\{1 - \mathrm{i}\epsilon N(\mu - 1 - \mathrm{i}\delta)\}\right] \\ &\quad + \mathrm{O}(\delta^2) + \mathrm{O}(N^2). \end{aligned} \tag{7.3}$$

Equation (7.3) is valid when μ is bounded away from zero, $|\mu| \geqslant \mu_0 > 0$ for some fixed μ_0. This condition is not troublesome as will be discussed below (7.4). The expression between square brackets in (7.3), the expansion of the square root, is correct upto third order in N for the real part, while the imaginary part is correct upto fourth order in N. We separate the real and imaginary part of (7.3) and neglect δ with respect to η and N^2 on account of (7.1). Then we find

$$\begin{aligned} R &= 2(\mu - 1)^2 + (m - \mu^4 D)|\mu|\left\{1 - \frac{N(\mu - 1)^2}{2\mu^2}\right\} + \mathrm{O}(N^2) \\ &\quad + \mathrm{i}\left\{(m - \mu^4 D)\frac{|\mu|}{2\mu^2}\epsilon N^2(\mu - 1)^3 + \eta|\mu| + \mathrm{O}(\eta N) + \mathrm{O}(N^3)\right\}. \end{aligned} \tag{7.4}$$

We are interested in Im R when Re $R=0$. Since Re $R \approx 2$ for $\mu=0$ the condition imposed on μ ($\mu \geqslant \mu_0 > 0$) is satisfied for the values of μ of interest. In the case of Re $R=0$, the corresponding values of μ must approximately satisfy

$$2(\mu-1)^2+(m-\mu^4 D)\,|\mu|=0. \tag{7.5}$$

The equation (7.5) corresponds with the equation for the wavelength (3.1) for the two-sided infinite plate. For values of μ satisfying (7.5), Im R becomes

$$\operatorname{Im} R=\eta\,|\mu|+\frac{\epsilon N^2(1-\mu)^5}{\mu^2}+\mathrm{O}(\eta N)+\mathrm{O}(N^3). \tag{7.6}$$

Analogous to Section 4 we can show that for m and D in region II[a] this expression is positive for all sufficiently small values of η and N. So we reach the conclusion that R does not encircle the origin when $\lambda=\mu-\mathrm{i}\delta$ moves for constant δ just below and parallel to the real axis from $\mu=\operatorname{Re}\lambda=-\infty$ towards $\mu=\operatorname{Re}\lambda=+\infty$.

Next we consider the factor S of R in (7.2),

$$S \stackrel{\text{def}}{=} \frac{\mu-\mathrm{i}\delta-\mathrm{i}q}{D\left\{(\mu-\mathrm{i}\delta)^2+s^2\right\}^{3/2}\left\{(\mu-\mathrm{i}\delta)^2+q^2\right\}(\mu-1-\mathrm{i}\delta)}. \tag{7.7}$$

The argument of the denominator of S increases by π when μ passes from $-\infty$ towards $+\infty$. Hence, if we take the constant q, introduced in (6.3), to be positive, the total change of the argument of (7.7) is zero when λ moves on L from Re $\lambda=-\infty$ towards Re $\lambda=+\infty$. Then also the argument of the product $-RS$ which is the function G of (7.2), will have the same value at both "ends" of the line L.

8. The polynomial $Q(\lambda)$

Because (6.12) is the one-sided Fourier transform of the velocity component in the y-direction in the wake of the plate, we have to require

$$\lim_{|\lambda|\to\infty}\frac{\partial\phi_-}{\partial y}(\lambda,0)=\lim_{|\lambda|\to\infty}\{H_-(\lambda)+Q(\lambda)\}X_-(\lambda)=0,\quad \lambda\in S^-. \tag{8.1}$$

Since we have from (6.6)

$$\lim_{|\lambda|\to\infty}G_\pm(\lambda)=1,\quad \lambda\in S^\pm, \tag{8.2}$$

the asymptotic behaviour of $X_-(\lambda)$, (6.7), is given by

$$\lim_{|\lambda|\to\infty}X_-(\lambda)=\frac{\mathrm{i}D}{2}\lambda(\lambda-\mathrm{i}s)^{1/2}+\mathrm{O}(|\lambda|^{1/2}),\quad \lambda\in S^-. \tag{8.3}$$

Hence the behaviour of $H_-(\lambda)$, (6.10), for $|\lambda| \to \infty$ is determined by

$$\lim_{|\lambda|\to\infty} H_-(\lambda) = P(\lambda) + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \mathrm{O}(|\lambda|^{-5/2}), \quad \lambda \in S^-, \tag{8.4}$$

where $P(\lambda)$ is a polynomial of degree three. The four coefficients of this polynomial and the constants a_1 and a_2 depend on m, D, $w(0)$, $\partial w/\partial x(0)$ and the values of

$$X_-\left(-\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right), \quad X_-\left(-\mathrm{i}\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right), \quad X_+\left(\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)$$

and $\quad X_+\left(\mathrm{i}\left(\frac{m+\mathrm{i}\eta}{D}\right)^{1/4}\right)$.

Since for given m and D the functions $X_-(\lambda)$ and $X_+(\lambda)$ are fixed, the six constants mentioned above are completely determined by the values m, D, $w(0)$ and $\partial w/\partial x(0)$.

In order to satisfy the condition (8.1) we have on account of (8.3) and (8.4) to choose the still unknown polynomial $Q(\lambda)$ as

$$Q(\lambda) = -P(\lambda). \tag{8.5}$$

Next we have to choose the unknown constants $w(0)$ and $\partial w/\partial x(0)$ such that

$$a_1 = 0, \tag{8.6}$$

which by (8.3) is necessary for satisfying (8.1). Then the asymptotic behaviour of (8.1) becomes

$$\lim_{|\lambda|\to\infty} \frac{\partial\phi_-}{\partial y}(\lambda, 0) = a_2 \frac{\mathrm{i}D}{2\lambda}\sqrt{\lambda - \mathrm{i}s} - \frac{w(0) - \mathrm{i}\dfrac{\partial w}{\partial x}(0)}{\sqrt{2\pi}\,\lambda} + \mathrm{O}(|\lambda|^{-3/2}), \quad \lambda \in S^-. \tag{8.7}$$

The leading term in (8.7) gives rise to a square-root singularity of the velocity in the y-direction at the trailing edge of the plate. This follows from the inverse Fourier transformation which yields the following asymptotic behaviour for $x \uparrow 0$:

$$\begin{aligned}
&\lim_{x\uparrow 0} \frac{a_2 D}{2\sqrt{2\pi}} \left\{ -\int_{-\infty}^{0} \frac{\mathrm{e}^{-\mathrm{i}\lambda x}}{\sqrt{|\lambda|}}\,\mathrm{d}\lambda + \mathrm{i}\int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}\lambda x}}{\sqrt{\lambda}}\,\mathrm{d}\lambda \right\} \\
&= \lim_{x\to 0} \frac{a_2 D}{2\sqrt{2\pi}} (\mathrm{i}-1) \int_0^\infty \frac{\cos\lambda|x| + \sin\lambda|x|}{\sqrt{\lambda}}\,\mathrm{d}\lambda \\
&= \lim_{x\to 0} \frac{a_2 D}{2\sqrt{2\pi}} (\mathrm{i}-1) \frac{2}{\sqrt{|x|}} \int_0^\infty (\cos u^2 + \sin u^2)\,\mathrm{d}u \\
&= \lim_{x\to 0} \frac{a_2 D(\mathrm{i}-1)}{2\sqrt{|x|}}.
\end{aligned} \tag{8.8}$$

Hence we have to require

$$a_2 = 0, \tag{8.9}$$

in order to satisfy the Kutta condition, or in other words to have finite velocities near the trailing edge of the plate.

9. The vorticity at the plate

In the preceding section we found a square-root singularity for the velocity near the trailing edge of the plate when $a_2 \neq 0$, (8.9). We will show that then the bound vorticity per unit of length in the $\bar{x}$-direction, $\bar{\Gamma}(\bar{x}, \bar{t})$, which replaces the plate, has also a square-root singularity. This vorticity is reckoned positive when $\bar{\Gamma}(\bar{x}, \bar{t})$ points with a right-hand screw in the positive $\bar{z}$-direction (Figure 2.1).

After introducing the dimensionless quantities (2.10) and $\Gamma(x)$ by

$$\bar{\Gamma}(\bar{x}, \bar{t}) = U\Gamma(x)\, \mathrm{e}^{-\mathrm{i}t}, \tag{9.1}$$

the equation of motion (2.13) of the plate can, in a close neighbourhood of the trailing edge, be written as

$$\Gamma(x) = -mw + D\frac{\partial^4 w}{\partial x^4} - \mathrm{i}\eta w, \tag{9.2}$$

where η is the damping coefficient of the plate. Here we neglected the term $\partial[\Phi]^+_-/\partial t$ of (2.3) because even if a square-root singularity of $\Gamma(x)$ occurs, this remains finite. So a square-root singularity of $\Gamma(x)$ corresponds to a square-root singularity of $\partial^4 w/\partial x^4$. By use of the solution $W_+(\lambda)$, (6.13), we will calculate this singularity of $\partial^4 w/\partial x^4$. Fourier transformation of $\partial^4 w/\partial x^4$ yields

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{\partial^4 w}{\partial x^4}\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x = \lambda^4 W_+(\lambda) + \frac{\lambda^2}{\sqrt{2\pi}}\frac{\partial w}{\partial x}(0) - \frac{\mathrm{i}\lambda^3}{\sqrt{2\pi}}w(0), \tag{9.3}$$

where, as in Section 5, we supposed

$$\frac{\partial^2 w}{\partial x^2}(0) = \frac{\partial^3 w}{\partial x^3}(0) = 0. \tag{9.4}$$

To determine the behaviour of $W_+(\lambda)$ for $|\lambda| \to \infty$, $\lambda \in S^+$, we must determine by (6.13) the asymptotic behaviour of $X_+(\lambda)$ and $H_+(\lambda)$. Using (6.7) we obtain

$$\lim_{|\lambda| \to \infty} X_+(\lambda) = \frac{1}{\lambda^2\sqrt{\lambda + \mathrm{i}s}} + \mathrm{O}(|\lambda|^{-7/2}), \quad \lambda \in S^+. \tag{9.5}$$

Combination of (6.10), (8.4), (8.5) and (8.6) yields

$$\lim_{|\lambda|\to\infty} -H_+(\lambda)+Q(\lambda) = \lim_{|\lambda|\to\infty}\left(\frac{w(0)\lambda^3 D + \mathrm{i}\lambda^2 D\dfrac{\partial w}{\partial x}(0)}{\sqrt{2\pi}}\right)\frac{-\mathrm{i}}{(m+\mathrm{i}\eta-\lambda^4 D)X_+(\lambda)}$$

$$+P(\lambda)+\frac{a_1}{\lambda}+\frac{a_2}{\lambda^2}+\mathrm{O}(|\lambda|^{-3})+Q(\lambda)$$

$$=\lim_{|\lambda|\to\infty}\frac{\mathrm{i}}{\lambda^4 X_+(\lambda)}\left(\frac{w(0)\lambda^3+\mathrm{i}\lambda^2\dfrac{\partial w}{\partial x}(0)}{\sqrt{2\pi}}\right)+\frac{a_2}{\lambda^2}$$

$$+\frac{\mathrm{i}w(0)}{\sqrt{2\pi}}\frac{(m+\mathrm{i}\eta)}{D}\frac{\sqrt{\lambda+\mathrm{i}s}}{\lambda^3}+\mathrm{O}(|\lambda|^{-3}),\quad \lambda\in S^+. \tag{9.6}$$

Hence we have

$$\lim_{|\lambda|\to\infty}\lambda^4 W_+(\lambda)+\frac{\lambda^2}{\sqrt{2\pi}}\frac{\partial w}{\partial x}(0)-\frac{\mathrm{i}\lambda^3}{\sqrt{2\pi}}w(0)$$

$$=\lim_{|\lambda|\to\infty}\frac{a_2}{\sqrt{\lambda+\mathrm{i}s}}+\frac{\mathrm{i}w(0)}{\sqrt{2\pi}}+\mathrm{O}(|\lambda|^{-3/2}),\quad \lambda\in S^+. \tag{9.7}$$

Using the inverse Fourier transformation we find the following asymptotic behaviour

$$\lim_{x\downarrow 0}\frac{\partial^4 w}{\partial x^4}=\lim_{x\downarrow 0}\frac{a_2}{\sqrt{2\pi}}\left(-\mathrm{i}\int_{-\infty}^{0}\frac{\mathrm{e}^{-\mathrm{i}\lambda x}}{\sqrt{|\lambda|}}\mathrm{d}\lambda+\int_0^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\lambda x}}{\sqrt{\lambda}}\mathrm{d}\lambda\right)=\lim_{x\to 0}\frac{a_2(1-\mathrm{i})}{\sqrt{|x|}}. \tag{9.8}$$

By (9.2) the result for the bound vorticity $\Gamma(x)$ is

$$\lim_{x\downarrow 0}\Gamma(x)=\lim_{x\downarrow 0}\frac{a_2 D(1-\mathrm{i})}{\sqrt{|x|}}. \tag{9.9}$$

This singularity of the bound vorticity induces indeed a singularity of the disturbance velocity in the y-direction of the type given in (8.8).

10. The waves in the plate

In this section we recapitulate the waves which occur in the semi-infinite plate in case $(m, D)\in \mathrm{II}^{\mathrm{a}}$.

The inverse Fourier transformation yields for the displacement $w(x)$ of the plate

$$w(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{-H_+(\lambda)+Q(\lambda)\}K_+(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}\lambda. \tag{10.1}$$

Hence the deformation pattern for $x \to +\infty$ is determined by the poles of

$$\{-H_+(\lambda)+Q(\lambda)\}X_+(\lambda), \tag{10.2}$$

lying just below the line of integration. We have (6.1)

$$X_-(\lambda)=X_+(\lambda)T(\lambda), \tag{10.3}$$

where $X_-(\lambda)$ and $X_+(\lambda)$ are regular functions without zeroes in the lower- and upper-halfplane, respectively, defined in (6.7). So the zeroes of $T(\lambda)$ with a positive imaginary part are zeroes of $X_-(\lambda)$, while the zeroes of $T(\lambda)$ with a negative imaginary part are poles of $X_+(\lambda)$. The pole $\lambda=1$ of $T(\lambda)$, which is also a pole of $X_-(\lambda)$, describes the periodic disturbance velocity in the wake, which has a non-dimensional length period 1.

In Section 4 we have discussed the zeroes of $T(\lambda)$ for $(m, D) \in \mathrm{II}^a$. First, we found one zero with a positive real part and a negative imaginary part. This corresponds to a strongly-increasing downstream wave. Second, we had a zero with a negative real part and, in the case of non-zero values of η and N, a negative imaginary part. This yields a slowly-decreasing upstream wave.

From (6.10) it is seen that the first factor of (10.2) also has poles with a negative imaginary part at

$$\lambda=-\left(\frac{m+i\eta}{D}\right)^{1/4}, \qquad \lambda=-i\left(\frac{m+i\eta}{D}\right)^{1/4}. \tag{10.4}$$

However, a simple calculation, using (6.1), shows that the residue belonging to these values of λ is zero. So they give no contribution to $w(x)$.

From the foregoing it follows that the wave pattern of the plate for large values of x, consists of a strongly-increasing downstream wave and a slowly-decreasing upstream wave. By this the "leading edge" at $x=+\infty$ is at rest for non-zero values of η and N.

11. Numerical results

In order to satisfy the Kutta condition at the trailing edge of the waving plate we have to satisfy the two conditions (8.6) and (8.9), namely

$$a_1=0, \qquad a_2=0. \tag{11.1}$$

These quantities are defined by (8.4) and can be written as

$$a_1=C_1w(0)+C_2\frac{\partial w}{\partial x}(0)=0, \tag{11.2}$$

$$a_2=C_3w(0)+C_4\frac{\partial w}{\partial x}(0)=0, \tag{11.3}$$

where $C_1, \ldots, C_4$ can be calculated as complex-valued functions of m and D ($\eta=0$) in a straight-forward way.

At first sight, to satisfy (11.2) and (11.3), it seems that we can take

$$w(0) = \frac{\partial w}{\partial x}(0) = 0. \tag{11.4}$$

However, then, by (6.10), $H_-(\lambda) = H_+(\lambda) = 0$ and, by (8.4) and (8.5), $Q(\lambda) = 0$. Hence by (6.12), $(\partial \Phi_-/\partial y)(\lambda, 0) = 0$ and no motion of the plate exists. From this we conclude that, in order to allow for a hydro-elastic motion, we have to take $(w(0), \partial w/\partial x(0)) \neq (0, 0)$.

But then it has to be required

$$Z(m, D) \overset{\text{def}}{=} |C_1C_4 - C_2C_3| = 0, \tag{11.5}$$

where the absolute value of the complex quantity is taken. It is not difficult to show that, although the values $C_1, \ldots, C_4$ depend on m and D in a "more general way", the zeroes of $Z(m, D)$ depend only on m/D.

The function $Z(m, D)$ contains some complicated integrals namely

$$X_-\left(-\mathrm{i}\left(\frac{m}{D}\right)^{1/4}\right), \qquad X_+\left(\mathrm{i}\left(\frac{m}{D}\right)^{1/4}\right), \tag{11.6}$$

which are defined by (6.7) and which have to be calculated numerically.

A check on the computer program is that the quotient (6.1),

$$X_-(\lambda)/X_+(\lambda) = T(\lambda), \tag{11.7}$$

has to be independent from the chosen values of q and s which occur in $X_-(\lambda)$ and $X_+(\lambda)$. We do not enter into the numerical details but refer to [5] (in Dutch).

As has been mentioned already several times, we confine ourselves to

$$(m, D) \in \mathrm{II}^{\mathrm{a}}. \tag{11.8}$$

It turns out that on the part inside of II^{a} of a circle around the origin, which encloses region IV and cuts the D-axis in a point with $D < 4$ (Figure 11.1), the function $Z(m, D) \neq 0$. Then, because the zeroes of $Z(m, D)$ depend only on m/D (below (11.5)),

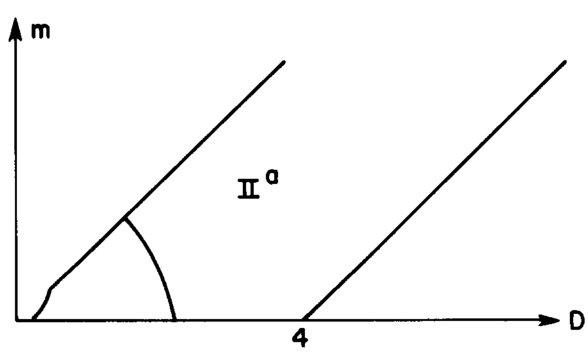

Figure 11.1. The region II^{a} with circle.

it follows that

$$Z(m, D) \neq 0, \qquad (m, D) \in \mathrm{II}^{\mathrm{a}}. \tag{11.9}$$

Hence, for this set of parameters, the Kutta condition at the trailing edge of the plate cannot be satisfied.

We conclude with a remark about what could have been the consequence of finding a zero $(m, D) = (m^*, D^*)$ of $Z(m, D)$. Suppose we have a physical situation defined by $\overline{m}$, $\overline{D}$, U and ρ. Then by (2.10) we find

$$\frac{m^*}{D^*} = \frac{\overline{m}U^4}{\overline{D}\omega^2}, \tag{11.10}$$

from which follows an ω for which the Kutta-condition is satisfied. Then by experiment it could be tried to verify that this ω occurs when the plate carries out its hydro-elastic motion.

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